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THE PROBLEM OF THE SOLID CYLINDER COMPRESSED BETWEEN ROUGH RIGID STAMPS

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Abstract—Since the attempt of Filon in 1902 to solve the title problem the solution then obtained was apparently not essentially improved. Present work offers such an improvement for all length–diameter ratio's larger than 0·1. The eigenfunction technique of Little and Childs is further developed into a method, such that the stress-singularities at the circumference of the end-planes can be incorporated. Presence of these singularities prevented some other methods to deliver reliable results.

1. INTRODUCTION

MUCH attention has been paid to the determination of the stress distribution in solid semi-infinite and finite circular cylinders with stress-free curved surfaces and with given stresses or given displacements at the plane ends. Apparently a satisfactory solution for the solid cylinder compressed between rough rigid stamps does not yet exist. This is the more remarkable since this kind of loading seems to be important for solid cylinders, while the mathematical formulation of the problem is, of course, most simple.

As early as 1902 Filon [1] gave an approximate solution by replacing the boundary condition for the radial displacements at the ends into a boundary condition for the radial displacements of the circumference of the end planes only and some reasonably looking assumption on the shear stresses at the end planes. In 1944 Pickett [2] gave a solution by using the Fourier method. To obtain the coefficients in the Fourier series he had to solve an infinite set of linear equations. The approximate solution of these equations did not lead to satisfactory results in the neighbourhood of the circumference of the end planes.

It is probable that more investigators have tried to obtain a solution for such an obvious problem. Whatever method is used, one has to solve an infinite set of linear equations and most attempts to solve such a set will have shown bad convergence. This is due to a stress-singularity at the circumference of the end planes, which affects the results for the stresses also in regions inside the cylinder. Some investigators will have been aware of the presence of such a singularity and its character, others perhaps not.

By the little known work of Knein [3] in 1927 and the work of Williams [4] attention has been drawn in literature to the existence and character of the singularities:

$$\rho \to 0: \sigma = \operatorname{Re} c\rho^{-a}, \qquad 0 < \operatorname{Re} a < 1 \tag{1.1}$$

where in this case σ is a stress along a plane end of the cylinder, ρ a distance from the circumference, a an exponent which is identified with the "character" of the singularity and c the coefficient which determines its "strength".

In the present work existing methods to deal with semi-infinite and finite solid cylinders were investigated as to the possibility to adapt them in such a way that for the compressed cylinder the stress-singularity and the associated convergence-difficulties could be conquered. All these methods require for all boundary conditions of practical importance the solution of an infinite set of equations. This is not necessary, if, and only if, the boundary conditions are such that the problem can be transformed into a problem for a cylinder infinite in length (to both sides).†

There are two important classes of methods:

1. Fourier analysis methods such as that of Pickett [2] and of Valov [5].
2. Solutions in the form of eigenfunction expansions. To obtain the participation factors in these expansions, Lurje [6] used the method of least squares, Schiff [7], Nuller [8] and Little and Childs [9] employed (bi) orthogonality relations between these eigenfunctions.‡

Further methods have been formulated by Horvay and Mirabel [11] and by Mendelson and Roberts [12].

From all these methods, which have proved to be successful in solving the problems with boundary conditions for which they were applied, the work of Little and Childs [9] was thought to be most suitable to start the development of the present method.

Of course the question what has been done previously for the corresponding plane-problem presents itself. In fact the work of Knein [3] concerns such a problem. Benthem [13], Miklowitz [14], Vorovich and Kopasenko [15], deal with semi-infinite clamped strips loaded in their plane in such a way that the stress-singularities in the corners are fully analyzed. However, the adaptation of these methods to the axisymmetric problem meets with great, if not insurmountable difficulties.

The present study was prompted by tests on cylindrical rock-pieces performed by the Mining Department of the University of Technology, Delft.

2. MATHEMATICAL FORMULATION

Both a semi-infinite cylinder, occupying the region $r \leq 1$, $0 \leq z < \infty$, and a finite cylinder, occupying the region $r \leq 1$, $-\frac{1}{2}L \leq z \leq \frac{1}{2}L$ will be considered. The cylinders are loaded axisymmetrically, without torsion. The normal stresses are $\sigma_r, \sigma_\phi, \sigma_z$ and τ_{rz} is the non-zero shear stress. The non-zero displacements in radial and axial direction are, respectively, u and w.

The curved surfaces of both cylinders are free of stresses:

$$r = 1 : \sigma_r = 0, \qquad \tau_{rz} = 0. \tag{2.1}$$

The conditions at the plane ends are:
for the semi-infinite cylinder

$$z = 0 : u = 0, \qquad w = 0, \tag{2.2a}$$

$$z = \infty : \sigma_z = -\sigma_{zg}, \qquad \tau_{rz} = 0; \tag{2.2b}$$

† Such boundary conditions for a plane end are the so called "mixed" ones, i.e. one displacement component and one stress component is prescribed for a plane end.

‡ Also in a recent paper [10] for a related cylinder problem such an eigenfunction expansion with its (bi) orthogonality properties was exploited. But, unfortunately, the desire to escape the necessity to solve an infinite set of equations led to an incorrect result (Appendix B).

and for the finite cylinder

$$z = L/2: u = 0, \qquad w = -\frac{1+\nu}{E} w_g, \tag{2.3a}$$

$$z = -L/2: u = 0, \qquad w = +\frac{1+\nu}{E} w_g, \tag{2.3b}$$

where σ_{zg} is a given constant stress, while w_g is a constant, such that at $z = \pm L/2$ holds

$$\int_0^1 \sigma_z r \,\mathrm{d}r = -\tfrac{1}{2}\sigma_{zg}.$$

The axisymmetric cylinder problem can be described with the aid of the two differential equations of Navier–Cauchy for the displacement components u and w. Introducing the Love strain function (ψ), these two equations may be reduced to one biharmonic equation for ψ (see for this reduction for instance Biezeno–Grammel [16]):

$$\nabla^2\nabla^2\psi = 0, \tag{2.4}$$

where ∇^2 is the Laplacian operator in cylindrical coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{2.5}$$

The nonzero stresses and displacements, expressed in terms of ψ are:

$$\sigma_r = \frac{\partial}{\partial z}\left\{\nu\nabla^2\psi - \frac{\partial^2\psi}{\partial r^2}\right\}, \tag{2.6a}$$

$$\sigma_\phi = \frac{\partial}{\partial z}\left\{\nu\nabla^2\psi - \frac{1}{r}\frac{\partial\psi}{\partial r}\right\}, \tag{2.6b}$$

$$\sigma_z = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2\psi - \frac{\partial^2\psi}{\partial z^2}\right\}, \tag{2.6c}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\psi - \frac{\partial^2\psi}{\partial z^2}\right\}, \tag{2.6d}$$

$$u = -\frac{1+\nu}{E}\frac{\partial^2\psi}{\partial r\,\partial z}, \tag{2.6e}$$

$$w = \frac{1+\nu}{E}\left\{2(1-\nu)\nabla^2\psi - \frac{\partial^2\psi}{\partial z^2}\right\}. \tag{2.6f}$$

With the aid of (2.6) the boundary conditions (2.1) and (2.2) or (2.1) and (2.3) can be expressed in terms of ψ. The biharmonic equation with these boundary conditions form the mathematical formulation of the problem.

3. A GENERAL SOLUTION FOR THE LOVE FUNCTION WHICH SATISFIES THE CONDITIONS AT THE CURVED SURFACES

A solution of (2.4) for the strain function of the form

$$\psi = M(\gamma, r)\,\mathrm{e}^{-\gamma z} \tag{3.1}$$

is tried.

Substitution in the biharmonic equation (2.4) yields:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\gamma^2\right)^2 M(\gamma, r) = 0. \tag{3.2}$$

The solution bounded at $r = 0$ is:

$$M(\gamma, r) = B_1(\gamma)J_0(\gamma r)+B_2(\gamma)rJ_1(\gamma r). \tag{3.3}$$

Satisfying the boundary conditions at the curved surfaces (2.1), two homogeneous linear equations for B_1 and B_2 are obtained. There will be only a non-zero solution when γ is a root of the equation:

$$\gamma^6\left[J_0^2(\gamma)+\left\{1-\frac{2(1-\nu)}{\gamma^2}\right\}J_1^2(\gamma)\right] = 0. \tag{3.4}$$

Indicating the solution for the strain function resulting from the 6-fold root $\gamma_0 = 0$ with a superscript h, it becomes:

$$\psi^h = b_1+b_2\ln r+b_3z+b_4r^2+b_5z^2+b_6\frac{1-2\nu}{6(1+\nu)}\left(\frac{3\nu}{1-2\nu}r^2z+z^3\right). \tag{3.5}$$

This function belongs to a homogeneous state of stress only. The related stresses and displacements are

$$\begin{aligned} &\sigma_r^h = \sigma_\phi^h = \tau_{rz}^h = 0; \qquad &\sigma_z^h = \sigma_0; \\ &u^h = -\nu\frac{\sigma_0}{E}r; \qquad &w^h = \frac{\sigma_0}{E}z+K, \end{aligned} \tag{3.6}$$

where $\sigma_0(= b_6)$ and $K\{= (1+\nu)[8b_4(1-\nu)+2b_5(1-2\nu)]/E\}$ are constants.

The non-zero complex roots of the transcendental equation (3.4) are written as $\gamma_j(j = 1,2\ldots)$. The absolute value of γ_j increases with increasing j. When γ_j is a root, then also $-\gamma_j$, $\bar{\gamma}_j$ and $-\bar{\gamma}_j$ are roots (a bar denotes the complex conjugate), i.e. for every j there are four roots. The asymptotic formula for the roots γ_j stems from Dougall [17] in 1914; he found for large n:

$$\gamma_n \sim n\pi-\frac{\ln(4n\pi)}{4n\pi}+\frac{i}{2}\ln(4n\pi). \tag{3.7}$$

For small values of j the roots γ_j were computed by Prokopov [18] in 1950. Later Little and Childs [9] gave those roots in more significant digits for some values of Poisson's ratio ν.

Indicating the solution for the strain function resulting from the non-zero roots γ_j with a superscript e, we have:

$$\psi^e = \sum_{j=1}^{\infty} a(\gamma_j)\left[-\left\{2(1-\nu)\frac{J_1(\gamma_j)}{\gamma_j}+J_0(\gamma_j)\right\}J_0(\gamma_jr)-J_1(\gamma_j)rJ_1(\gamma_jr)\right]\mathrm{e}^{-\gamma_jz}. \tag{3.8}$$

Analogous to the approach of Little and Childs [9] a 4-vector $\mathbf{f}$, related to stresses and displacements, is expanded in a series of 4-vectors of eigenfunctions:

$$\mathbf{f} = \left[\sigma_z^e, \tau_{rz}^e, \frac{E}{1+\nu}u^e, \frac{E}{1+\nu}w^e\right] = \sum_{j=1}^{\infty} a(\gamma_j)\boldsymbol{\phi}(\gamma_j, r)\,\mathrm{e}^{-\gamma_jz}. \tag{3.9}$$

A 4-vector $\boldsymbol{\phi}(\gamma_j, r)$ of eigenfunctions consists of the following components:

$$\phi^{(1)}(\gamma_j, r) = [2\gamma_j^2 J_1(\gamma_j) - \gamma_j^3 J_0(\gamma_j)]J_0(\gamma_j r) - \gamma_j^3 J_1(\gamma_j) r J_1(\gamma_j r), \tag{3.10a}$$

$$\phi^{(2)}(\gamma_j, r) = \gamma_j^3 J_1(\gamma_j) r J_0(\gamma_j r) - \gamma_j^3 J_0(\gamma_j) J_1(\gamma_j r), \tag{3.10b}$$

$$\phi^{(3)}(\gamma_j, r) = -\gamma_j^2 J_1(\gamma_j) r J_0(\gamma_j r) + [2(1-\nu)\gamma_j J_1(\gamma_j) + \gamma_j^2 J_0(\gamma_j)] J_1(\gamma_j r), \tag{3.10c}$$

$$\phi^{(4)}(\gamma_j, r) = [\gamma_j^2 J_0(\gamma_j) - 2(1-\nu)\gamma_j J_1(\gamma_j)] J_0(\gamma_j r) + \gamma_j^2 r J_1(\gamma_j) J_1(\gamma_j r). \tag{3.10d}$$

The following remarks are to be made on this eigenfunction expansion:

1. From the analysis of Little and Childs [9] may be concluded that indeed (3.8) is an expansion in a complete set of eigenfunctions when it is possible to describe the stresses and the displacements at the flat ends with the aid of Fourier–Bessel expansions or with the aid of Dini's series of Besselfunctions.
2. It is important to realize that, in principle, the summations in (3.8) and (3.9) have to be done over the eigenvalues in the whole complex γ-plane; so every j represents in fact four terms, though, at least at the semi-infinite cylinder, finally the eigenfunctions stemming from one half of the complex γ-plane will be eliminated. This summation over the whole complex plane is done in contrast with other approaches (like in [9], where the roots with $\mathrm{Re}(\gamma_j) < 0$ are eliminated immediately).
3. The stresses acting at a cross-section, which are to be described with the aid of the eigenfunction expansion (3.8), are self-equilibrating:

$$\int_0^1 \sigma_z^e r \, \mathrm{d}r = 0. \tag{3.11}$$

4. The (bi)orthogonality relations of the functions $\phi^{(i)}(\gamma_j, r)$ are insufficient to use Fourier methods for analyzing the participation factors in a direct manner.

A general solution for the strain function in axialsymmetric problems on cylinders with stress-free curved surfaces is the sum of (3.5) and (3.8). The unknown constants in this solution have to be obtained with the aid of the conditions at the plane ends.

4. THE PARTICIPATION FACTORS $a(\gamma_j)$ WHEN THE VECTOR f IS KNOWN AT A CROSS-SECTION

It would be possible to get an expression for the participation factors $a(\gamma_j)$ in (3.8) when at a cross-section (say at $z = l$) all components of the vector $\mathbf{f} = \mathbf{f}_b(r)$ should be known. Then (3.9) may be written as

$$\mathbf{f}_b(r) = \sum_{j=1}^{\infty} a(\gamma_j)\boldsymbol{\phi}(\gamma_j, r)\,\mathrm{e}^{-\gamma_j l}. \tag{4.1}$$

Little and Childs [9] developed a set of functions $W^{(i)}(\gamma_j, r)$ $(i = 1, 2, 3, 4)$ biorthogonal† to the functions $\phi^{(i)}(\gamma_j, r)$ when $\mathrm{Re}(\gamma_j) > 0$. It is easy to check that their results are also valid when $\mathrm{Re}(\gamma_j) < 0$. So for all eigenvalues γ_j holds the biorthogonality relation:

$$\int_0^1 \mathbf{W}(\gamma_k, r)\cdot\boldsymbol{\phi}(\gamma_j, r) r \, \mathrm{d}r = 0 \qquad \text{if } \gamma_j \neq \gamma_k, \tag{4.2a}$$

$$= N(\gamma_j) \quad \text{if } \gamma_j = \gamma_k, \tag{4.2b}$$

† Older work on the general theory for the development of vectors of eigenfunctions and their biorthogonal counterparts is that of Birkhoff and of Langer (1939) and is discussed in some length by Kamke in his book [19]. For cylinder-problems Schiff [7] (1883) found a generalized orthogonality relation.

where

$$N(\gamma_j) = (1-\nu)[-4\gamma_j J_0^2(\gamma_j) - 2\gamma_j J_1^2(\gamma_j) + 4(1-\nu)J_0(\gamma_j)J_1(\gamma_j)]. \quad (4.2c)$$

It be emphasized that in general the notation γ_j refers to the four roots of (3.4) with equal absolute values:

$$\gamma_{j,1} = \xi_j + i\eta_j, \quad \gamma_{j,2} = \xi_j - i\eta_j, \quad \gamma_{j,3} = -\xi_j + i\eta_j, \quad \gamma_{j,4} = -\xi_j - i\eta_j \ (\xi \text{ and } \eta \text{ real});$$

however, with condition (4.2a) is meant that $\gamma_j \neq \gamma_k$ is also fulfilled for:

$$\gamma_j = \xi_j + i\eta_j \quad \text{and} \quad \gamma_k = \pm\xi_j - i\eta_j \text{ or } \gamma_k = -\xi_j + i\eta_j.$$

The biorthogonal vector $\mathbf{W}(\gamma_j, r)$ has the following components:

$$W^{(1)}(\gamma_j, r) = \{[\gamma_j J_0(\gamma_j) - 2(1-\nu)J_1(\gamma_j)]J_0(\gamma_j r) + \gamma_j J_1(\gamma_j) r J_1(\gamma_j r)\}/\{2\gamma_j J_1^2(\gamma_j)\}, \quad (4.3a)$$

$$W^{(2)}(\gamma_j, r) = \{\gamma_j r J_1(\gamma_j) J_0(\gamma_j r) - [2(1-\nu)J_1(\gamma_j) + \gamma_j J_0(\gamma_j)]J_1(\gamma_j r)\}/\{2\gamma_j J_1^2(\gamma_j)\}, \quad (4.3b)$$

$$W^{(3)}(\gamma_j, r) = \{-\gamma_j^2 r J_1(\gamma_j) J_0(\gamma_j r) + \gamma_j^2 J_0(\gamma_j) J_1(\gamma_j r)\}/\{2\gamma_j J_1^2(\gamma_j)\}, \quad (4.3c)$$

$$W^{(4)}(\gamma_j, r) = \{[2\gamma_j J_1(\gamma_j) - \gamma_j^2 J_0(\gamma_j)]J_0(\gamma_j r) - \gamma_j^2 J_1(\gamma_j) r J_1(\gamma_j r)\}/\{2\gamma_j J_1^2(\gamma_j)\}. \quad (4.3d)$$

By use of the biorthogonality relation now all the participation factors indeed are obtained:

$$a(\gamma_j) = \frac{e^{\gamma_j l}}{N(\gamma_j)} \int_0^1 \{W^{(1)}(\gamma_j, r) f_b^{(1)}(r) + W^{(2)}(\gamma_j, r) f_b^{(2)}(r) + W^{(3)}(\gamma_j, r) f_b^{(3)}(r)$$
$$+ W^{(4)}(\gamma_j, r) f_b^{(4)}(r)\}\, r\, dr. \quad (4.4)$$

5. THE SEMI-INFINITE CYLINDER

The solution given in Section 3 satisfies the boundary conditions (2.1). The boundary conditions (2.2) are now resolved into two conditions for the general solutions (3.6) and (3.8):

$$z = 0: u \equiv u^e + u^h = 0, \qquad w \equiv w^e + w^h = 0; \quad (5.1)$$

$$z = \infty: \sigma_z \equiv \sigma_z^e + \sigma_z^h = -\sigma_{zg}, \qquad \tau_{rz} \equiv \tau_{rz}^e + \tau_{rz}^h = 0. \quad (5.2)$$

From (5.2) and (3.11) follows the homogeneous solution:

$$\sigma_z^h = -\sigma_{zg}; \qquad \sigma_r^h = \sigma_\phi^h = \tau_{rz}^h = 0;$$
$$u^h = \frac{\nu}{E}\sigma_{zg} r; \qquad w^h = -\frac{\sigma_{zg}}{E} z + \frac{1+\nu}{E} K, \quad (5.3)$$

where K is an unknown constant, such that the eigenfunction expansion (3.8) is possible.

Our problem is now reduced to: find the participation factors $a(\gamma_j)$ in the eigenfunction expansion (3.8) and the constant K such that the following conditions are satisfied:

$$z = 0: \frac{E}{1+\nu} u^e = -\frac{\nu}{1+\nu}\sigma_{zg} r, \qquad \frac{E}{1+\nu} w^e = -K; \quad (5.4)$$

$$z \to \infty: \psi^e \to 0. \quad (5.5)$$

However in no cross-section all the four components of the vector **f** are known, so that the participation factors cannot be obtained directly from expression (4.4). Yet it is possible to use this expression when we describe the lacking components $\overset{0}{\sigma_z^e}$ and τ_{rz}^e at $z = 0$ with the aid of series. In doing so, it must be realized that there may occur a stress singularity in the corner at ($z = 0, r = 1$). Along the lines of Appendix A it is concluded that the stresses become infinite exponentially when the distance from the corner tends to zero. So the stresses in the corner can be described with great accuracy by

$$\sigma_z = -c_1\rho^{-\lambda} \quad \text{and} \quad \tau_{rz} = c_2\rho^{-\lambda} \quad \text{when } \rho \to 0$$

where λ is a constant with $0 < \lambda < 1$ satisfying equation (A.9), while the constants c_1 and c_2 are related by equation (A.11). The series for the stresses at $z = 0$ are now formed in such a way that the stress singularities are described accordingly. For $z = 0, 0 \le r < 1$ we take:

$$\sigma_z^e = \sigma_z - \sigma_z^h = -c_1\{(1-r)^{-\lambda} + \lambda(1-r)\} + \sigma_{zg} - \sum_{m=1}^{\infty} b_m J_0(\mu_m r), \tag{5.6a}$$

$$\tau_{rz}^e = \tau_{rz} - \tau_{rz}^h = c_2\{(1-r)^{-\lambda} - (1-r) - \tfrac{1}{2}\lambda(\lambda+1)(r^2 - r)\} + \sum_{n=1}^{\infty} d_n J_1(\kappa_n r), \tag{5.6b}$$

where

$$c_2 = -\frac{-\lambda - 1 + 2\nu}{-\lambda + 2 - 2\nu}\,\mathrm{tg}\left(\lambda\frac{\pi}{2}\right)\cdot c_1, \tag{5.6c}$$

$$\mu_m \text{ is the } m\text{th root of } J_0(\mu) = 0,$$

$$\kappa_n \text{ is the } n\text{th root of } J_1(\kappa) = 0.$$

In this way the unknown functions are expressed in terms of the unknown coefficients c_1, b_m, d_n. Some remarks about these expressions have to be made.

1. When $r = 0$ there may occur discontinuities in the third and higher derivatives of σ_z^e and in the fourth and higher derivatives of τ_{rz}^e. It is possible to describe the stresses with functions which are everywhere continuously differentiable by another assumption for the terms with c_1 and c_2. Yet the description (5.6) is chosen, for this will lead to simpler calculations.
2. The infinite series in (5.6) are chosen as Fourier–Bessel series to simplify the calculations.
3. It is also possible to describe the stresses σ_z^e and τ_{rz}^e with Fourier–Bessel expansions only, though in a very slowly convergent way. The slow convergence is caused by the stress singularities. In description (5.6) the singularities are eliminated from the infinite number of terms of the Fourier–Bessel expansion. This has the advantage that the remaining series in (5.6) are rapidly convergent. However, the set of functions used here becomes overcomplete when all the infinite terms of the Fourier–Bessel expansions are considered. This will cause restrictions to the solution to be obtained.
4. Expansion (5.6a) has to satisfy condition (3.11). This leads to one condition between the constants c_1 and b_m.

With the aid of the series expansions (5.6) for the stresses at $z = 0$ and with the given displacements (5.4) at $z = 0$ a vector $\mathbf{f}_b(r)$ may be composed. After substitution in (4.4) the participation factors $a(\gamma_j)$ are known in terms of the unknown constants c_1, b_m, d_n:

$$a(\gamma_j) = \frac{1}{N(\gamma_j)}\int_0^1 \left\{W^{(1)}(\gamma_j, r)(\sigma_z^e)_{z=0} + W^{(2)}(\gamma_j, r)(\tau_{rz}^e)_{z=0} - \frac{\nu}{1+\nu}\sigma_{zg} r W^{(3)}(\gamma_j, r)\right\} r\,\mathrm{d}r. \tag{5.7}$$

Note that in (5.7) the term with $W^{(4)}$ drops because

$$\int_0^1 W^{(4)}(\gamma_j, r) r\,\mathrm{d}r = 0.$$

These relations between $a(\gamma_j)$ and the unknown constants are valid for both $\mathrm{Re}(\gamma_j) > 0$ and $\mathrm{Re}(\gamma_j) < 0$. However the remaining condition $\psi^e \to 0$ for $z \to \infty$ (5.5) requires:

$$a(\gamma_j) = 0 \quad \text{if } \mathrm{Re}(\gamma_j) < 0. \tag{5.8}$$

The conditions (5.7) and (5.8) deliver the set of infinite equations for the unknown coefficients in the series expansions for the stresses at $z = 0$:

$$\int_0^1 \left\{(\sigma_z^e)_{z=0} W^{(1)}(\gamma_j, r) + (\tau_{rz}^e)_{z=0} W^{(2)}(\gamma_j, r) - \frac{\nu}{1+\nu}\sigma_{zg} r W^{(3)}(\gamma_j, r)\right\} r\,\mathrm{d}r = 0$$

$$\text{if } \mathrm{Re}(\gamma_j) < 0 \quad (j = 1, 2, 3, \ldots), \tag{5.9}$$

while

$$\int_0^1 (\sigma_z^e)_{z=0} r\,\mathrm{d}r = 0.$$

[Remember, every j in (5.9) delivers two equations arising from $\gamma_j = \xi_j \pm i\eta_j$.] Using the series expansions (5.6) for the stresses at $z = 0$, the integrals in (5.9) resulting from the singularities in the stresses are to be developed into usual power series or in asymptotic series. The other integrals are easily to be computed in a direct manner.

The infinite set of equations (5.9) in c_1, b_m, d_n is solved by taking only a finite number of unknowns, putting the others zero. Calculations are performed for Poisson's ratio $\nu = 0{\cdot}25$. The number of unknowns was taken successively 3, 5, 7, 9 and 11. In the last case the unknowns were $c_1, b_1, d_1, b_2, d_2, b_3, d_3, b_4, d_4, b_5, d_5$. Using these solutions in the series (5.6) the stresses $\sigma_z (= \sigma_z^e + \sigma_z^h)$ and $\tau_{rz} (= \tau_{rz}^e + \tau_{rz}^h)$ at $z = 0$ are computed for $0 \leq r \leq 1$. The successive approximations are given in Tables 1 and 2. The convergence to a final result is obvious. Figures 1 and 2 give graphs of the stresses σ_z and τ_{rz} at $z = 0$; they are based on the best values of Tables 1 and 2, respectively.

As stated earlier the stresses at $z = 0$ (5.6) are described with the aid of overcomplete sets of functions. When the number of unknowns in the procedure of solving the infinite set of equations is not great, then the linear independency of the terms with the stress singularity and the Fourier–Bessel terms ensures rapid convergence to the results. Taking the number of unknowns infinite, then the matrix of coefficients of the equations (5.9) will be singular, owing to the overcompleteness of the set of functions taken in (5.6). So the matrix of coefficients becomes more and more ill-conditioned by taking the number of unknowns in the procedure of solving the infinite set of equations higher and higher. Then computer-computations become impossible; the convergence of the results breaks down. This explains why not many unknowns are tried. Besides, in view of the results already obtained, this is not necessary.

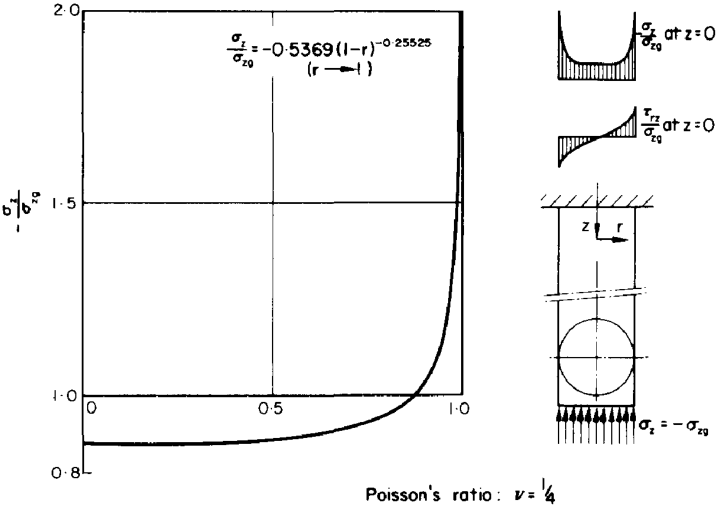

FIG. 1. The stress ratio σ_z/σ_{zg} at $z = 0$.

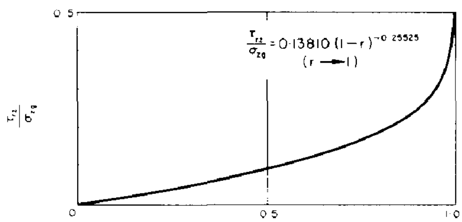

FIG. 2. The stress ratio τ_{rz}/σ_{zg} at $z = 0$.

TABLE 1. THE STRESS RATIOS σ_z/σ_{zg} AT $z = 0$; $\nu = \frac{1}{4}$; $\lambda = 0.25525$

r	No. of unknowns used				
	3	5	7	9	11
+0.000	−0.9253	−0.8689	−0.8881	−0.8874	−0.8770
+0.200	−0.9165	−0.8779	−0.8845	−0.8856	−0.8870
+0.400	−0.8960	−0.8949	−0.8872	−0.8868	−0.8872
+0.600	−0.8821	−0.9044	−0.9038	−0.9025	−0.9011
+0.800	−0.9264	−0.9374	−0.9432	−0.9452	−0.9463
+0.900	−1.0426	−1.0355	−1.0364	−1.0371	−1.0372
+0.980	−1.5146	−1.4820	−1.4740	−1.4715	−1.4706
+0.990	−1.8017	−1.7605	−1.7499	−1.7465	−1.7454
+0.995	−2.1475	−2.0972	−2.0839	−2.0797	−2.0783
+0.999	−3.2357	−3.1587	−3.1383	−3.1317	−3.1295
+1.000	$-0.55482(1-r)^{-\lambda}$	$-0.54160(1-r)^{-\lambda}$	$-0.53808(1-r)^{-\lambda}$	$-0.53694(1-r)^{-\lambda}$	$-0.53657(1-r)^{-\lambda}$

TABLE 2. THE STRESS RATIOS τ_{rz}/σ_{zg} AT $z = 0$; $\nu = \frac{1}{4}$; $\lambda = 0{\cdot}25525$

r	No. of unknowns used				
	3	5	7	9	11
+0·000	+0·0000	+0·0000	+0·0000	+0·0000	+0·0000
+0·200	+0·0436	+0·0253	+0·0431	+0·0361	+0·0348
+0·400	+0·0872	+0·0709	+0·0717	+0·0782	+0·0794
+0·600	+0·1333	+0·1318	+0·1245	+0·1203	+0·1203
+0·800	+0·1930	+0·1966	+0·1993	+0·1999	+0·1988
+0·900	+0·2460	+0·2453	+0·2478	+0·2504	+0·2526
+0·980	+0·3852	+0·3771	+0·3756	+0·3756	+0·3761
+0·990	+0·4612	+0·4508	+0·4483	+0·4478	+0·4479
+0·995	+0·5513	+0·5384	+0·5351	+0·5342	+0·5340
+0·999	+0·8320	+0·8122	+0·8070	+0·8053	+0·8048
+1·000	+0·14271 $(1-r)^{-\lambda}$	+0·13930 $(1-r)^{-\lambda}$	+0·13840 $(1-r)^{-\lambda}$	+0·13810 $(1-r)^{-\lambda}$	+0·13791 $(1-r)^{-\lambda}$

From the results in Tables 1 and 2 it is ascertained that the ratio $|\tau_{rz}/\sigma_z|$ at $z = 0$ ensure indeed the condition $w = 0$ provided the coefficient of friction between stamp and cylinder is greater than 0.257 [being the c_2/c_1 ratio of (5.6c) and (A.11)].

The participation factors $a(\gamma_j)$ may now be computed with the aid of (5.7) for all eigenvalues γ_j with $\mathrm{Re}(\gamma_j) > 0$, while $a(\gamma_j) = 0$ for all eigenvalues γ_j with $\mathrm{Re}(\gamma_j) < 0$. It is also possible to find the constant K with the aid of the eigenfunction expansion for w^e at $z = 0$. Subsequently all the stresses (when $z \neq 0$) and all the displacements in the cylinder may be computed with the aid of the homogeneous state of stress and the eigenfunction expansions. Also these series showed very good convergency to final values.

6. THE FINITE CYLINDER

The solution given in Section 3 satisfies the boundary conditions (2.1). Splitting up the boundary conditions (2.3) to adapt them to the solutions (3.6) and (3.8) learns that the homogeneous solution, which satisfies the symmetry with regard to $z = 0$, is:

$$\sigma_z^h = -\sigma_{zg}; \qquad \sigma_r^h = \sigma_\phi^h = \tau_{rz}^h = 0;$$
$$u^h = \frac{\nu}{E}\sigma_{zg} r; \; w^h = -\frac{\sigma_{zg}}{E} z \tag{6.1}$$

and that the participation factors $a(\gamma_j)$ in the eigenfunction expansion (3.8) and the constant w_g must now follow from the conditions:

$$z = +\frac{L}{2}: \frac{E}{1+\nu} w^e = -w_g + \frac{\sigma_{zg}}{1+\nu}\tfrac{1}{2}L, \qquad \frac{E}{1+\nu} u^e = -\frac{\nu}{1+\nu}\sigma_{zg} r. \tag{6.2a}$$

$$z = -\frac{L}{2}: \frac{E}{1+\nu} w^e = w_g - \frac{\sigma_{zg}}{1+\nu}\tfrac{1}{2}L, \qquad \frac{E}{1+\nu} u^e = -\frac{\nu}{1+\nu}\sigma_{zg} r. \tag{6.2b}$$

As in Section 5 the stresses σ_z^e and τ_{rz}^e at $z = -L/2, 0 \leq r \leq 1$ are described with the aid of the series expansions (5.6). This description has no sense when the cylinder is very short. It is known by the principle of De St.-Venant that in that case there is hardly any influence of the curved surface on the solution for $r < \frac{1}{2}D - L$, where D is the diameter of the cylinder. Consequently the stresses σ_z^e and τ_{rz}^e must be almost constant in that region and the terms with the stress singularities in (5.6) do not form any more the proper first approximation. Good results were only obtained when the length–diameter ratio was greater than 0·1.

After the composition of the vector $\mathbf{f}_b(r)$ at $z = -L/2$ and substitution in (4.4) the participation factors in terms of c_1, b_m, d_n become:

$$a(\gamma_j) = \frac{e^{-\gamma_j L/2}}{N(\gamma_j)} \int_0^1 \left\{ (\sigma_z^e)_{z=-L/2} W^{(1)}(\gamma_j, r) + (\tau_{rz}^e)_{z=-L/2} W^{(2)}(\gamma_j, r) - \frac{\nu}{1+\nu} \sigma_{zg} r W^{(3)}(\gamma_j, r) \right\} r \, \mathrm{d}r. \tag{6.3}$$

Again these relations are valid for both $\mathrm{Re}(\gamma_j) > 0$ and $\mathrm{Re}(\gamma_j) < 0$.

The symmetry of the cylinder with regard to the plane $z = 0$, requires:

$$a(\gamma_j) = -a(-\gamma_j). \tag{6.4}$$

To satisfy both (6.3) and (6.4) we have again a system of infinite equations for the unknown coefficients in the series expansions for the stresses at $z = -L/2$. Using the relation $N(\gamma_j) = -N(-\gamma_j)$ these equations are:

$$\begin{aligned} e^{-\gamma_j L/2} \int_0^1 & \left\{ (\sigma_z^e)_{z=-L/2} W^{(1)}(\gamma_j, r) + (\tau_{rz}^e)_{z=-L/2} W^{(2)}(\gamma_j, r) - \frac{\nu}{1+\nu} \sigma_{zg} r W^{(3)}(\gamma_j, r) \right\} r \, \mathrm{d}r \\ = e^{\gamma_j L/2} \int_0^1 & \left\{ (\sigma_z^e)_{z=-L/2} W^{(1)}(-\gamma_j, r) + (\tau_{rz}^e)_{z=-L/2} W^{(2)}(-\gamma_j, r) \right. \\ & \left. - \frac{\nu}{1+\nu} \sigma_{zg} r W^{(3)}(-\gamma_j, r) \right\} r \, \mathrm{d}r, \qquad j = 1, 2, 3 \ldots \end{aligned} \tag{6.5}$$

while $\int_0^1 \sigma_z^e r \, \mathrm{d}r = 0$.

[Remember, every j in (6.5) delivers two equations].

As in Section 5 the unknown coefficients in the series expansions (5.6) for the stresses at $z = -L/2$ may now be solved. In Fig. 3 the results for the strength of the singularity (c_1) is shown in dependency of the length–diameter ratio of the cylinder when $\nu = \frac{1}{4}$.

After the solution of the unknowns c_1, b_m and d_n, the participation factors $a(\gamma_j)$ may be computed with the aid of (6.3) if $\mathrm{Re}(\gamma_j) > 0$ and with (6.4) if $\mathrm{Re}(\gamma_j) < 0$. Subsequently the stresses and displacements in all points of the cylinder may follow. In the calculations which were performed, again very good convergency was observed.

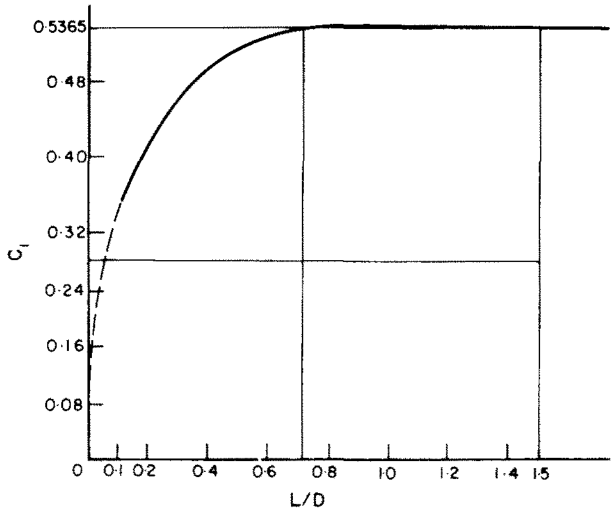

FIG. 3. The strength of the singularity c_1 as a function of the length–diameter ratio L/D; $\nu = \frac{1}{4}$.

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APPENDIX A

The stress singularity at the circumference of the plane surface of the cylinder

The Navier–Cauchy equations for the displacement components of the cylinder are

$$(2-2\nu)\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}\right\}+(1-2\nu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right) = 0, \tag{A.1a}$$

$$(2-2\nu)\frac{\partial}{\partial z}\left\{\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}\right\}-\frac{(1-2\nu)}{r}\frac{\partial}{\partial r}\left\{r\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right)\right\} = 0. \tag{A.1b}$$

The homogeneous boundary conditions of the cylinder along the sides of the corner $z = 0, r = 1$ are

$$z = 0, \qquad u = 0, \tag{A.2a}$$

$$w = 0, \tag{A.2b}$$

$$r = 1, \qquad \sigma_r = 0 \quad \text{or } (1-\nu)\frac{\partial u}{\partial r}+\nu\frac{u}{r}+\nu\frac{\partial w}{\partial z} = 0, \tag{A.2c}$$

$$\tau_{rz} = 0 \quad \text{or } \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r} = 0. \tag{A.2d}$$

In the corner $z = 0, r = 1$ a local polar coordinate system ρ, θ is introduced with the relations

$$r = 1-\rho\cos\theta, \tag{A.3a}$$

$$z = \rho\sin\theta, \tag{A.3b}$$

$$\frac{\partial}{\partial r} = -\cos\theta\frac{\partial}{\partial\rho}+\frac{1}{\rho}\sin\theta\frac{\partial}{\partial\theta}, \tag{A.3c}$$

$$\frac{\partial}{\partial z} = \sin\theta\frac{\partial}{\partial\rho}+\frac{\cos\theta}{\rho}\frac{\partial}{\partial\theta}, \tag{A.3d}$$

$$u = -u_\rho\cos\theta+u_\theta\sin\theta, \tag{A.3e}$$

$$w = u_\rho\sin\theta+u_\theta\cos\theta, \tag{A.3f}$$

$$\frac{1}{r} = 1+\rho\cos\theta+\rho^2\cos^2\theta+\rho^3\cos^3\theta+\ldots, \tag{A.3g}$$

where u_ρ and u_θ are the displacement components in the polar coordinate system.

Together with the introduction of (A.3) into (A.1), (A.2) a solution for u_ρ and u_θ, at least valid for small values of ρ, is tried in the form

$$u_\rho = \sum_n u_{\rho,n}, \qquad u_\theta = \sum_n u_{\theta,n}, \qquad n = 0, 1, 2, 3\ldots \tag{A.4}$$

$$u_{\rho,n} = \rho^{1-\lambda+n}f_n(\theta), \qquad u_{\theta,n} = \rho^{1-\lambda+n}g_n(\theta). \tag{A.5}$$

We separate those parts of the equations which concern u_ρ and u_θ with the lowest powers in ρ, i.e. $\rho^{1-\lambda}$ and have the equations

$$(2-2\nu)\frac{\partial}{\partial\rho}\left\{\frac{1}{\rho}\frac{\partial(\rho u_{\rho,0})}{\partial\rho}+\frac{1}{\rho}\frac{\partial u_{\theta,0}}{\partial\theta}\right\}-\frac{1-2\nu}{\rho^2}\frac{\partial}{\partial\theta}\left\{\frac{\partial(\rho u_{\theta,0})}{\partial\rho}-\frac{\partial u_{\rho,0}}{\partial\theta}\right\}=0, \tag{A.6a}$$

$$(2-2\nu)\frac{1}{\rho^2}\frac{\partial}{\partial\theta}\left\{\frac{\partial(\rho u_{\rho,0})}{\partial\rho}+\frac{\partial u_{\theta,0}}{\partial\theta}\right\}+(1-2\nu)\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\left\{\frac{\partial(\rho u_{\theta,0})}{\partial\rho}-\frac{\partial u_{\rho,0}}{\partial\theta}\right\}\right]=0, \tag{A.6b}$$

$$\theta=0,\qquad u_{\rho,0}=0, \tag{A.7a}$$

$$u_{\theta,0}=0, \tag{A.7b}$$

$$\theta=\frac{\pi}{2},\qquad \frac{\partial u_{\rho,0}}{\partial\rho}+\frac{1-\nu}{\rho}\left(u_{\rho,0}+\frac{\partial u_{\theta,0}}{\partial\theta}\right)=0, \tag{A.7c}$$

$$\frac{\partial u_{\rho,0}}{\rho\partial\theta}+\frac{\partial u_{\theta,0}}{\partial\rho}-\frac{u_{\theta,0}}{\rho}=0. \tag{A.7d}$$

But (A.6), (A.7) are exactly the homogeneous equations for the wedge in plane strain, one side ($\theta=0$) fixed, one side ($\theta=\pi/2$) stress free. This problem was solved by Knein [3] and later by Williams [4].

We put

$$u_{\rho,0}+iu_{\theta,0}=\rho^{1-\lambda}A\{(3-4\nu)\,e^{i\theta(1-\lambda)}-(1-\lambda)\,e^{i\theta(1+\lambda)}\}+\rho^{1-\lambda}B\,e^{-i\theta(1-\lambda)} \tag{A.8}$$

which has the shape (A.5), $n=0$, and satisfies (A.6). The boundary conditions (A.7) deliver homogeneous equations for the complex constants A and B and λ must be a root of

$$\cos^2\left(\lambda\frac{\pi}{2}\right)=\frac{4(1-\nu)^2}{3-4\nu}-\frac{(1-\lambda)^2}{3-4\nu}. \tag{A.9}$$

The stresses σ_θ and $\tau_{\rho\theta}$ from the displacements $u_{\rho,0}$ and $u_{\theta,0}$ are

$$\sigma_\theta=\frac{2G}{1-2\nu}\left\{\nu\frac{\partial u_{\rho,0}}{\partial\rho}+\frac{1-\nu}{\rho}\left(u_{\rho,0}+\frac{\partial u_{\theta,0}}{\partial\theta}\right)\right\}, \tag{A.10a}$$

$$\tau_{\rho\theta}=G\left(\frac{\partial u_{\rho,0}}{\rho\partial\theta}+\frac{\partial u_{\theta,0}}{\partial\rho}+\frac{u_{\theta,0}}{\rho}\right) \tag{A.10b}$$

and they have the exponents $-\lambda$.

Along $\theta=0(z=0)\sigma_\theta=\sigma_z$, $\tau_{\rho\theta}=-\tau_{rz}$ and their ratio is

$$\left(\frac{\tau_{\rho\theta}}{\sigma_\theta}\right)_{\substack{\theta=0\\ \rho\to 0}}=-\left(\frac{\tau_{rz}}{\sigma_z}\right)_{\substack{z=0\\ r\to 1}}=\frac{\lambda+1-2\nu}{-\lambda+2-2\nu}\operatorname{tg}\left(\lambda\frac{\pi}{2}\right). \tag{A.11}$$

A root λ from (A.11) with real part greater than 1 would lead to infinite strain-energy in the corner region (and infinite resultants of stresses). The gravest root which has to be regarded is ($\nu=\frac{1}{4}$)

$$\lambda=0{\cdot}25525, \tag{A.12}$$

the next gravest root $\lambda=-0{\cdot}718\pm 0{\cdot}474i$ is considered to be of little interest for the series developments of (5.6).

A solution $u_{\rho,0}$; $u_{\theta,0}$ (for certain λ) once obtained should allow to compute (with much effort) a solution for $u_{\rho,1}$; $u_{\theta,1}$ in the development (A.4), (A.5) and so on. The exponent of the stresses stemming from $u_{\rho,1}$; $u_{\theta,1}$ is, however, only $-\lambda+1$ and is also of no interest for our purpose.

APPENDIX B

Comment on the paper by W. Flügge and V. S. Kelkar [10]

Flügge and Kelkar deal with a semi-infinite cylinder, which has no prescribed (zero) stresses along its curved surface, but prescribed (zero) displacements and which has prescribed displacements along its plane end. With that case they demonstrate their recommended general method for semi-infinite cylinders.

Also the cylinder with zero-displacements along its curved boundary has its 4-vectors of eigenfunctions with eigenvalues γ_j and their counterparts, the 4-vectors of biorthogonal functions which are derived by the authors in the proper way. It may be then possible to develop an arbitrary set of 4 functions (which are of bounded variation) in the range $0 < r < 1$ in an expansion of the eigenfunctions.

Also the authors of [10] have struggled with the lack of two boundary conditions at the plane end of the semi-infinite cylinder, only two boundary conditions being known. These latter conditions are notated here as

$$\begin{aligned} u &= f_1(r) \\ w &= f_2(r), \end{aligned} \tag{B.1}$$

u and w are displacement-components in radial and axial directions and f_1 and f_2 are known functions.

The lacking boundary conditions for two differential-expressions of the displacements u and w, say $D_3(u, w)$ and $D_4(u, w)$, are not expressed with the aid of series with unknown parameters as is done in our Sections 5 and 6, but in the equations for $D_3(u, w)$ and $D_4(u, w)$ arbitrarily two zero right-hand sides are chosen

$$\begin{aligned} D_3(u, w) &= 0 \\ D_4(u, w) &= 0. \end{aligned} \tag{B.2}$$

An expansion of the set of functions $f_1(r), f_2(r), 0, 0$ in vectors of eigenfunctions follows.

In this expansion only half of the eigenvalues are used. The other half was intentionally omitted because these would lead to infinite stresses at infinity.

Generally to fulfill four conditions for $u, w, D_3(u, w), D_4(u, w)$ like in (B.1) and (B.2) *all* eigenvalues with their 4-vectors of functions are necessary. It cannot be proved that for an arbitrary choice for the right-hand sides of (B.2) only the use of half of the eigenvalues with their 4-vectors of functions would be sufficient (it should be remembered that the omitted 4-vectors of functions are linearly independent of the used ones).

Thus it cannot be proved that the expansions obtained (with half of the eigenvalues) deliver indeed the result

$$\begin{aligned} u &= f_1(r) \\ w &= f_2(r) \\ D_3(u, w) &= 0 \\ D_4(u, w) &= 0 \end{aligned} \tag{B.3}$$

or even only (which would even suffice)

$$\begin{aligned} u &= f_1(r) \\ w &= f_2(r) \\ D_3(u, w) &= f_3(r) \\ D_4(u, w) &= f_4(r) \end{aligned} \tag{B.4}$$

where f_3 and f_4 are some non-zero functions. Neither are results like (B.3) or (B.4) demonstrated in a numerical example.

It is true that in our work the underlying principle involves a similar expansion in eigenfunctions with only half of the eigenvalues, but following that principle one should take for the right-hand sides of (B.2) the *actual* ones, which, however, are not known in advance. Such right-hand sides [of (5.6)] we provided with initially unknown parameters which allowed these right-hand sides to become the actual functions (after the solution of an infinite set of equations). Only for the actual right-hand sides of (B.2) it can be proved that the expansion with only half of the eigenfunctions delivers the desired result.

Indeed it is explained in [10] that the obtained eigenfunction expansion is not unique and depends on the chosen right hand sides of (B.2), but it was believed that the state of stress it represents is nevertheless unique, but clearly this is not proved for the region of the prescribed boundary conditions itself.

The leading term of the expansion for the stresses is also not unique. At some distance from the finite end (of the order of some diameters) this leading term will overwhelm the sum of all the others and so the state of stress is not unique there as well.

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Абстракт—Одиннадцать тонкостенных алюминиевых цилиндров с прямоугольными вырезами подвергали испытанию на сжатие по оси. При чем вырезы семи образцов под испытанием усилили различным армированием. Результаты испытания сравнивали с полученными результатами при критической продольной нагрузке цилиндра без армирования вырезов и с предсказанными вычислительной машиной нагрузками разрушения.

При таких тонкостенных цилиндрах, испытания и анализ вычислительной машины показывают, что усиление небольших и средних вырезов обычно бесполезно, т.к. это помогает только в случаях, если цилиндры сконструированы из чрезвычайно высококачественного материала. Обсуждаются сравнительные достоинства различных форм усилений для высококачественных цилиндров с различными вырезами усиление которых целесообразно и предлагается эмперический базис конструкции.